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CARTAN SUBALGEBRAS OF SIMPLE LIE ALGEBRAS([1])

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ABSTRACT. Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$. Let H be a Cartan subalgebra of L, let $L = H + \Sigma_{\gamma\in\Gamma}L_\gamma$ be the Cartan decomposition of L with respect to H, and let $\overline{H}$ be the restricted subalgebra of Der L generated by ad H. Let T denote the maximal torus of $\overline{H}$ and I denote the nil radical of $\overline{H}$. Then $\overline{H} = T + I$. Consequently, each $\gamma \in \Gamma$ is a linear function on H.

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F. Let H be a Cartan subalgebra of L and let $L = H + \Sigma_{\gamma\in\Gamma}L_\gamma$ be the Cartan decomposition of L with respect to H. The main results on the structure of H, that H is abelian and that ad h is semisimple for each $h \in H$, which hold when F is of characteristic zero are known [3, Satz 12] to fail when F is of prime characteristic. The resulting lack of information about the structure of H has been a severe handicap in the development of structure and classification theory for finite-dimensional simple Lie algebras of prime characteristic.

In this paper we assume that F is of characteristic $p > 7$ and investigate the structure of $\overline{H}$, the restricted subalgebra of Der L generated by ad H. Our main result (Theorem 2.1) is that $\overline{H} = T + I$ where T is the maximal torus of $\overline{H}$ and I is the nil radical of $\overline{H}$. (See §1 for definitions.) An immediate consequence (Corollary 2.2) is that each $\gamma \in \Gamma$ is a linear function on H. (Although in characteristic zero this is a trivial consequence of Lie's Theorem, the result is new in prime characteristic.)

This type of result was first proved by Schue [1] under the additional hypotheses that dim $T = 1$ and that every proper subalgebra of L is solvable.

Our proof begins with Schue's observation that if $\overline{H} \neq T + I$ then there exists $b \in \overline{H}$, $b \notin T + I$ such that $b^p \in I$ and $[b, \overline{H}] \subseteq T + I$. We then let

$$S = \{(\gamma, \delta) \in \Gamma \times \Gamma | \gamma([b[L_\delta, L_{-\delta}]]) \neq (0)\}.$$

Using Schue's techniques we show (§3) that there exist $\alpha, \beta \in \Gamma$ with $(\alpha, \beta) \in S$ and $(\beta, \alpha) \in S$. The argument then divides into two cases depending on

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whether or not there exists $\gamma \in \Gamma$ with $(\gamma, \gamma) \in S$. We consider these two cases separately (§§4 and 5), showing that either one leads to a contradiction.

These results have been announced in [4].

1. Preliminaries.

(1.1) If $K \supseteq L$ are Lie algebras let $N_K(L)$ denote the normalizer of L in K and let $C(K)$ denote the center of K.

Let R be a restricted Lie algebra over a field F of characteristic $p > 0$. If X is a subset of R let $\langle X \rangle$ denote the restricted subalgebra of R generated by X. If X is a subalgebra then $\langle X \rangle$ is clearly the F-span of $\{x^{p^i} | x \in X, i \geqslant 0\}$.

(1.2) Following [2, Chapter V.7] or [1, §1] we say that $x \in R$ is *semisimple* if $x \in \langle x^p \rangle$ and that $x \in R$ is *nilpotent* if $x^{p^n} = 0$ for some n. See also [5].

An ideal $J \subseteq R$ is said to be *nil* if every $x \in J$ is nilpotent. It is easily seen that a finite-dimensional restricted Lie algebra R contains a nil ideal I which contains every nil ideal. We call I the *nil radical* of R. An abelian subalgebra $T \subseteq R$ is called a *torus* if every element of T is semisimple.

(1.3) PROPOSITION. *Let R be a finite-dimensional restricted Lie algebra over a perfect field F. Then*:

(i) *If $x, y \in R$ are nilpotent (respectively, semisimple) and $[x, y] = 0$ then every element of $\langle\{x, y\}\rangle$ is nilpotent (respectively, semisimple).*

(ii) *If $x \in R$ then there exist $x_s, x_n \in \langle x \rangle$ such that x_s is semisimple, x_n is nilpotent, and $x = x_s + x_n$.*

(iii) *If $x, y, z \in R$, y semisimple, z nilpotent, $[y, z] = 0$, and $x = y + z$, then $y = x_s$ and $z = x_n$.*

(iv) *If R is nilpotent and $x \in R$ is semisimple then $x \in C(R)$.*

(v) *If R is nilpotent $\{x_s | x \in R\}$ is the unique maximal torus of R.*

PROOF. Parts (i)–(iii) are proved in Chapter V.7 of [2]. If x is semisimple then $x \in \langle x^{p^n} \rangle$ for any $n \geqslant 1$. Since R is nilpotent, $\mathrm{ad}(x^{p^n}) = (\mathrm{ad}\, x)^{p^n} = 0$ for sufficiently large n. Hence $\mathrm{ad}\, x = 0$, proving part (iv). Since $\{x_s | x \in R\}$ contains every semisimple element it must contain every torus of R. By (i) and (iv) $\{x_s | x \in R\}$ is a torus, proving part (v).

(1.4) Assume that F is algebraically closed, that L is a finite-dimensional simple Lie algebra over F, that H is a Cartan subalgebra of L, and that $L = H + \Sigma_{\gamma \in \Gamma} L_\gamma$ is the corresponding Cartan decomposition.

Identify L with the isomorphic subalgebra $\mathrm{ad}\, L$ of the restricted Lie algebra $\mathrm{Der}\, L$. Let $\bar{H} = \langle H \rangle \subseteq \mathrm{Der}\, L$. (Thus $\bar{H}$ is the F-span of $\{h^{p^n} | h \in H, n \geqslant 0\}$, where we write h^{p^n} for $(\mathrm{ad}\, h)^{p^n}$.) Let $\bar{L} = \bar{H} + L$.

LEMMA. (i) *$\bar{H}$ is a Cartan subalgebra of $\bar{L}$.*

(ii) *If $\bar{L} = \bar{H} + \Sigma_{\bar{\gamma} \in \bar{\Gamma}} \bar{L}_{\bar{\gamma}}$ is the Cartan decomposition of $\bar{L}$ with respect to $\bar{H}$ then the map $\bar{\gamma} \mapsto \bar{\gamma}|_H$ is a bijection of $\bar{\Gamma}$ onto Γ and $\bar{L}_{\bar{\gamma}} = L_{\bar{\gamma}|_H}$ for all $\bar{\gamma} \in \bar{\Gamma}$.*

PROOF. Since $[x, y^p] = x(\mathrm{ad}\, y)^p$ and since $\bar{H}$ is the F-span of $\{h^{p^n} | h \in H, n > 0\}$, we see by induction on i that $\bar{H}^i = H^i$ for all $i > 2$. Hence $\bar{H}$ is nilpotent. If $x \in L \cap N_{\bar{L}}(\bar{H})$ then $[x, H] \subseteq \bar{H} \cap L = H$. Thus $x \in N_L(H) = H$. Hence $\bar{H} = N_{\bar{L}}(\bar{H})$, proving (i).

The map $\bar{\gamma} \mapsto \bar{\gamma}|_H$ is clearly surjective, since for each $\gamma \in \Gamma$, $\bar{H}$ acts on L_γ and hence $L_\gamma \subseteq \sum_{\bar{\gamma} \in \bar{\Gamma}, \bar{\gamma}|_H = \gamma} \bar{L}_{\bar{\gamma}}$.

If $\bar{\gamma} \in \bar{\Gamma}$ then there exists $x \in \bar{H}$ such that $\bar{\gamma}(x) \neq 0$. Then ad x: $\bar{L}_{\bar{\gamma}} \to \bar{L}_{\bar{\gamma}}$ is surjective so $\bar{L}_{\bar{\gamma}} = [x, \bar{L}_{\bar{\gamma}}] \subseteq [\bar{L}, \bar{L}] \subseteq L$. Thus $\bar{L}_{\bar{\gamma}} \subseteq L_{\bar{\gamma}|_H}$.

Now suppose $\bar{\gamma}, \bar{\eta} \in \bar{\Gamma}$ and $\bar{\gamma}|_H = \bar{\eta}|_H \in \Gamma$. Then $\bar{L}_{\bar{\gamma}} \otimes (\bar{L}_{\bar{\eta}})^* \subseteq L_{\bar{\gamma}|_H} \otimes (L_{\bar{\eta}|_H})^*$ (where V^* denotes the contragredient module to V). Now H acts on $L_{\bar{\gamma}|_H} \otimes (L_{\bar{\eta}|_H})^*$. Moreover, the only weight is $\bar{\gamma}|_H - \bar{\eta}|_H = 0$. Hence H acts as a Lie algebra of nilpotent linear transformations on $L_{\bar{\gamma}|_H} \otimes (L_{\bar{\eta}|_H})^*$. By Engel's Theorem $\bar{H}$ also acts as a Lie algebra of nilpotent linear transformations on $L_{\bar{\gamma}|_H} \otimes (L_{\bar{\eta}|_H})^*$. Thus the only weight of $\bar{H}$ on $L_{\bar{\gamma}|_H} \otimes (L_{\bar{\eta}|_H})^*$ is 0. Since $\bar{\gamma} - \bar{\eta}$ is a weight of $\bar{H}$ on $\bar{L}_{\bar{\gamma}} \otimes (\bar{L}_{\bar{\eta}})^*$ we have $\bar{\gamma} = \bar{\eta}$. Thus the map $\bar{\gamma} \mapsto \bar{\gamma}|_H$ is bijective and hence $\bar{L}_{\bar{\gamma}} = L_{\bar{\gamma}|_H}$.

(1.5) In view of Lemma 1.4(ii) we may identify $\bar{\Gamma}$ and Γ and write $\bar{L} = \bar{H} + \sum_{\gamma \in \Gamma} L_\gamma$. Let T denote the maximal torus of $\bar{H}$ and I denote the nil radical of $\bar{H}$.

LEMMA. *Each $\gamma \in \Gamma$ is linear on $T + I$.*

PROOF. Since T is a torus each $t \in T$ acts diagonally on each L_γ and hence γ is linear on T.

If $h \in \bar{H}$ then by Lemma 1.3(ii) $h = h_s + h_n$. Then $h^{p^i} = h_s^{p^i} + h_n^{p^i}$ for all i, hence $h^{p^n} = h_s^{p^n}$ for sufficiently large n. Now if $y \in L_\gamma$ then

$$0 = y(\mathrm{ad}\, h - \gamma(h))^{p^n} = y(\mathrm{ad}\, h_s - \gamma(h))^{p^n}$$

for sufficiently large n. Hence $\gamma(h) = \gamma(h_s)$ for all $h \in H$. Now Lemma 1.3 shows that if $h \in T + I$ then $h_s \in T$, $h_n \in I$, and h_s is a linear function of h. Hence $\gamma(h) = \gamma(h_s)$ is a linear function of h.

(1.6) LEMMA. *Let X be a subset of H and $E = \{\gamma \in \Gamma | \gamma(X) \neq (0)\}$. If $E \neq \emptyset$ then $H = \sum_{\gamma \in E} [L_\gamma, L_{-\gamma}]$.*

PROOF. This is a special case of (4.2) of [1]. (The hypothesis in (4.2) of [1] that L is semirestricted can be dropped here, since we assume $X \subseteq H$.)

2. Statement of results. Our main result is

(2.1) THEOREM. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 7$. Let H be a Cartan subalgebra of L, $\bar{H}$ be the restricted subalgebra of* Der L *generated by* ad H, *T be the maximal torus of $\bar{H}$, and I be the nil ideal of $\bar{H}$. Then $\bar{H} = T + I$.*

(2.2) Let $L = H + \Sigma_{\gamma\in\Gamma} L_\gamma$ be the Cartan decomposition of L with respect to H. By Lemma 1.5 each $\gamma \in \Gamma$ is linear on $T + I$. By Theorem 2.1 $\overline{H} = T + I$. Hence we have

COROLLARY. *Each $\gamma \in \Gamma$ is a linear function on H.*

3. Action of $\overline{H}$ on L.

(3.1) Let F, L, and H be as in Theorem 2.1 and assume $\overline{H} \neq T + I$. We will eventually derive a contradiction, thus proving Theorem 2.1.

We begin with an analysis of the structure of $\overline{H}$ modeled on §3.5 of [1].

LEMMA. *If $\overline{H} \neq T + I$ then there exists $b \in \overline{H}$ such that*
(i) $b \notin T + I$,
(ii) $[b, \overline{H}] \subseteq T + I$,
(iii) $[b[\overline{H}, \overline{H}]] \subseteq I$,
(iv) $[b, x^p] \in I$ *for all* $x \in \overline{H}$,
(v) $b^p \in I$, *and*
(vi) $[b, H] \not\subseteq I$.

PROOF. Since T is central and I is an ideal, $T + I$ is a proper ideal of $\overline{H}$. Then $\overline{H}/(T + I)$ is a nonzero $\overline{H}$-module. As $\overline{H}$ is nilpotent, there exists $b \in \overline{H}$, $b \notin T + I$, such that $[b, \overline{H}] \subseteq T + I$. Since $b = b_s + b_n$ and $b_s \in T$ we see that $b_n \in \overline{H}$, $b_n \notin T + I$. Thus we may assume that $b = b_n$ is nilpotent. Now

$$[b^p, \overline{H}] = [b, \overline{H}](\text{ad } b)^{p-1} \subseteq (T + I)(\text{ad } b)^{p-1} \subseteq I$$

so $b^pF + I$ is a nil ideal containing I. Thus by the maximality of I, $b^p \in I$. Now,

$$[b[\overline{H}, \overline{H}]] \subseteq [[b, \overline{H}]\overline{H}] \subseteq [T + I, \overline{H}] \subseteq I$$

and

$$[b, x^p] = [b, x](\text{ad } x)^{p-1} \in (T + I)(\text{ad } x)^{p-1} \subseteq I$$

for all $x \in \overline{H}$. If $[b, H] \subseteq I$ then $[b, \overline{H}] \subseteq I$, and so $bF + I$ is a nil ideal containing I. This contradicts the maximality of I, so $[b, H] \not\subseteq I$.

(3.2) We continue to assume that L has Cartan decomposition $L = H + \Sigma_{\gamma\in\Gamma} L_\gamma$.

LEMMA. *There exists $\gamma \in \Gamma$ such that* $\gamma([b, H]) \neq (0)$.

PROOF. If $t \in T$ then $[y, t] = \gamma(t)y$ for all $y \in L_\gamma$. Hence $\gamma(t) = 0$ implies $[L_\gamma, t] = 0$ and $\gamma(t) = 0$ for all $\gamma \in \Gamma$ implies $t \in C(L) = (0)$.

Now if $t \in T$, $n \in I$ then we have seen (in (1.5)) that $\gamma(t + n) = \gamma(t)$. Thus $\gamma(t + n) = 0$ for all $\gamma \in \Gamma$ implies $t + n \in I$. Since, by Lemma 3.1 (vi), $[b, H] \not\subseteq I$, we have $\gamma([b, H]) \neq (0)$ for some $\gamma \in \Gamma$.

(3.3) Let $S = \{(\gamma, \delta) \in \Gamma \times \Gamma \mid \gamma([b[L_\delta, L_{-\delta}]]) \neq (0)\}$.

PROPOSITION. *Either*

$$(\alpha, \alpha) \in S \quad \textit{for some } \alpha \in \Gamma, \tag{3.3.1}$$

or

$$\textit{for all } \gamma \in \Gamma, (\gamma, \gamma) \notin S \textit{ but there exist } \alpha, \beta \in \Gamma \textit{ with } (\alpha, \beta) \in S \textit{ and } (\beta, \alpha) \in S. \tag{3.3.2}$$

PROOF. By Lemma 3.2 there exist $x \in H$ and $\gamma \in \Gamma$ such that $\gamma([b, x]) \neq 0$. Then by Lemma 1.6

$$H = \sum_{\gamma \in \Gamma,\, \gamma([b, x]) \neq 0} [L_\gamma, L_{-\gamma}].$$

Now by Lemma 3.1(vi) we have $[b, H] \not\subseteq I$. Thus there exists $\alpha \in \Gamma$ such that $\alpha([b, x]) \neq 0$ and $[b[L_\alpha, L_{-\alpha}]] \not\subseteq I$. Hence for some root δ, $\delta([b[L_\alpha, L_{-\alpha}]]) \neq (0)$. Again by Lemma 1.6 we have

$$H = \sum_{\gamma \in \Gamma,\, \gamma([b[L_\alpha, L_{-\alpha}]]) \neq (0)} [L_\gamma, L_{-\gamma}].$$

Since $x \in H$ and $\alpha([b, x]) \neq 0$ there exists $\beta \in \Gamma$ such that $\beta([b[L_\alpha, L_{-\alpha}]]) \neq (0)$ and $\alpha([b[L_\beta, L_{-\beta}]]) \neq (0)$. Thus either (3.3.1) or (3.3.2) holds.

(3.4) For $\gamma \in \Gamma$ define $\bar{H}_\gamma = \{x \in \bar{H} \mid \gamma([b, x]) = 0\}$. By Lemma 1.5 $\bar{H}_\gamma$ is a subspace of $\bar{H}$. By (iii) and (iv) of Lemma 3.1, $\bar{H}_\gamma$ is a restricted ideal of $\bar{H}$. If $\bar{H} \neq \bar{H}_\gamma$ fix an element $c_\gamma \in \bar{H}$ with $\gamma([b, c_\gamma]) = 1$.

Let V be a finite-dimensional irreducible restricted $\bar{H}$-module. Then VI is a submodule and, since I is nil, $VI \neq V$. Thus $VI = (0)$. Assume that $vt = \gamma(t)v$ for all $v \in V$, $t \in T$. Let $V_0 = \{v \in V \mid vb = 0\}$. Let $\{v_1, \ldots, v_n\}$ be a basis for V_0. Let $C = C_\gamma$.

LEMMA. (i) *For each k, $0 \leq k \leq p - 1$, the F-span of $\{v_i c^j \mid 1 \leq i \leq n, 0 \leq j \leq k\}$ is an $\bar{H}_\gamma$-subspace of V.*

(ii) $\{v_i c^j \mid 1 \leq i \leq n, 0 \leq j \leq p - 1\}$ *is a basis for V.*

PROOF. Since $T \cap \ker \gamma$ is an ideal of $\bar{H}$ of codimension 1 in T it is sufficient to prove the result under the assumption $\dim T = 1$. This is done in §3.7 of [1].

(3.5) Fix $\alpha \in \Gamma$ as in Lemma 3.3. For $\gamma \in \Gamma$ define a bilinear form on $L_\gamma \times L_{-\gamma}$ by

$$(x, y) = \alpha([b[x, y]]) \quad \text{for } x \in L_\gamma, y \in L_{-\gamma}.$$

In view of Lemma 3.1(iii) and the Jacobi identity we have

$$([x, h], y) = (x, [h, y]) \quad \text{for all } x \in L_\gamma, y \in L_{-\gamma}, h \in \bar{H}.$$

If X is an ad $\bar{H}$ invariant subspace of L_γ then $X^\perp = \{y \in L_{-\gamma} | (X, y) = (0)\}$ is an ad $\bar{H}$ invariant subspace of $L_{-\gamma}$. Define $K_{-\gamma} = L_\gamma^\perp$ for all $\gamma \in \Gamma$ and define $n_\gamma = \dim L_\gamma / K_\gamma$.

Now if $(\alpha, \alpha) \in S$ and if for some i, $1 \leqslant i \leqslant p - 1$, we have $n_{i\alpha} \neq 0$, then $(i\alpha, i\alpha) \in S$. Thus, replacing α by $i\alpha$ if necessary, we may assume that the root $\alpha \in \Gamma$ in (3.3.1) satisfies

$$n_\alpha \geqslant n_{i\alpha} \quad \text{for all } i, 1 \leqslant i \leqslant p - 1. \tag{3.5.1}$$

Similarly, if (3.3.2) holds and if $n_{\beta + i\alpha} \neq 0$ for some i, $0 \leqslant i \leqslant p - 1$, then we have $(\alpha, \beta + i\alpha) \in S$ and $(\beta + i\alpha, \alpha) \in S$. Hence, replacing β by $\beta + i\alpha$ if necessary, we may assume that the pair (α, β) in (3.3.2) satisfies

$$n_\beta \geqslant n_{\beta + i\alpha} \quad \text{for all } i, 0 \leqslant i \leqslant p - 1. \tag{3.5.2}$$

(3.6) We will complete the proof of Theorem 2.1 in the next two sections by showing that either conclusion in Lemma 3.3 leads to a contradiction. In §4 we will show that if (3.3.1) holds then $n_{2\alpha} + n_{3\alpha} > 2n_\alpha$, contradicting (3.5.1), and in §5 we will show that if (3.3.2) holds then $n_{\beta+\alpha} + n_{\beta-\alpha} \not> 2n_\beta$, contradicting (3.5.2).

4. Dimension arguments. I.

(4.1) We continue to let F, L, and H be as in Theorem 2.1 and to assume $\bar{H} \neq T + I$. In addition, we assume that (3.3.1) holds. Our object is to show that $n_{2\alpha} + n_{3\alpha} > 2n_\alpha$, thus contradicting (3.5.1).

(4.2) Let $L_\alpha \supseteq N_\alpha \supseteq M_\alpha \supseteq K_\alpha$, where K_α is as in (3.5), N_α and M_α are ad $\bar{H}$ invariant subspaces of L_α, and N_α / M_α is an irreducible $\bar{H}$-module (necessarily restricted).

For $X = M$ or N and for $i = 2, 3$ define

$$X'_{i\alpha} = \{x \in L_{i\alpha} | x(\operatorname{ad} L_{-\alpha})^{i-1} \subseteq X_\alpha\}$$

and

$$X_{i\alpha} = X'_{i\alpha} + K_{i\alpha}.$$

Then $X_{i\alpha}$ is an ad $\bar{H}$ submodule of $L_{i\alpha}$ and $L_{i\alpha} \supseteq N_{i\alpha} \supseteq M_{i\alpha} \supseteq K_{i\alpha}$ for $i = 2, 3$. By the Jacobi identity, $[X_\alpha, X_\alpha] \subseteq X_{2\alpha}$ and $[[X_\alpha, X_\alpha]X_\alpha] \subseteq X_{3\alpha}$.

LEMMA. $[X_\alpha^\perp, X_\alpha^\perp] \subseteq X_{2\alpha}^\perp$ *and* $[[X_\alpha^\perp, X_\alpha^\perp]X_\alpha^\perp] \subseteq X_{3\alpha}^\perp$.

PROOF.

$$\begin{aligned}([X_\alpha^\perp, X_\alpha^\perp], X_{2\alpha}) &\subseteq ([X_\alpha^\perp, X_\alpha^\perp], X'_{2\alpha}) + ([X_\alpha^\perp, X_\alpha^\perp], K_{2\alpha}) \\ &= ([X_\alpha^\perp, X_\alpha^\perp], X'_{2\alpha}) \subseteq (X_\alpha^\perp, [X_\alpha^\perp, X'_{2\alpha}]) \\ &\subseteq (X_\alpha^\perp, [L_{-\alpha}, X'_{2\alpha}]) \subseteq (X_\alpha^\perp, X_\alpha) = (0)\end{aligned}$$

so $[X_\alpha^\perp, X_\alpha^\perp] \subseteq X_{2\alpha}^\perp$. The other result is similar.

(4.3) For $v \in L$ write vC for $[v, c_\alpha]$. Let $V_0 = \{v \in N_\alpha | [v, b] \in M_\alpha\}$. (Thus

$V_0/M_\alpha = (N_\alpha/M_\alpha)_0$ in the notation of (3.4).) Choose $v_1, \ldots, v_n \in N_\alpha$ so that $\{v_1 + M_\alpha, \ldots, v_n + M_\alpha\}$ is a basis for V_0/M_α. Define $V_{i+1} = V_i + V_iC$ for $0 \leqslant i \leqslant p-2$. Then by Lemma 3.4 each V_i, $0 \leqslant i \leqslant p-1$, is an ad $\overline{H}_\alpha$ subspace of L_α, $N_\alpha = V_{p-1}$, and N_α/M_α has basis $\{v_iC^j + M_\alpha | 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant p-1\}$.

Thus

(4.3.1) $$\left[V_j, \overline{H}_\alpha\right] \subseteq V_j \quad \text{for } 0 \leqslant j \leqslant p-1$$

and, since $\overline{H} = c_\alpha F + \overline{H}_\alpha$,

(4.3.2) $$\left[V_j, \overline{H}\right] \subseteq V_{j+1} \quad \text{for } 0 \leqslant j \leqslant p-2.$$

Taking annihilators gives

(4.3.3) $$L_{-\alpha} \supseteq M_\alpha^\perp \supseteq V_0^\perp \supseteq V_1^\perp \supseteq \cdots \supseteq V_{p-1}^\perp = N_\alpha^\perp \supseteq K_{-\alpha},$$ $$\left[V_j^\perp, \overline{H}_\alpha\right] \subseteq V_j^\perp \quad \text{for } 0 \leqslant j \leqslant p-1$$

and

(4.3.4) $$\left[V_j^\perp, \overline{H}\right] \subseteq V_{j-1}^\perp \quad \text{for } 1 \leqslant j \leqslant p-1.$$

(4.4) Since $\overline{H} = c_\alpha F + \overline{H}_\alpha$, if $x \in L_\beta, y \in L_{-\beta}$ we have $[x,y] = c_\alpha u + h$ for some $u \in F$, $h \in \overline{H}_\alpha$. Then $(x,y) = \alpha([b[x,y]]) = u$. Hence we have

(4.4.1) $$[x,y] \in c_\alpha(x,y) + \overline{H}_\alpha \quad \text{for all } x \in L_\beta, y \in L_{-\beta}.$$

(4.5) Choose $d_{ij} \in M_\alpha^\perp$ for $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant p-1$, so that

(4.5.1) $$(v_rC^s, d_{ij}) = \delta_{ir}\delta_{js}.$$

Then $V_j^\perp/N_\alpha^\perp$ has basis $\{d_{ik} + N_\alpha^\perp | 1 \leqslant i \leqslant n, j < k \leqslant p-1\}$.

Let W denote the linear span of $\{v_iC^j | 1 \leqslant i \leqslant n, 1 < j < p-1\}$. Define

$$\Phi\colon W \wedge W \to [W, W] \subseteq [N_\alpha, N_\alpha] \subseteq N_{2\alpha}$$

by

$$(w_1 \wedge w_2)\Phi = [w_1, w_2].$$

Define

$$\Psi\colon W \wedge W \to [[W, W]N_\alpha] \subseteq [[N_\alpha, N_\alpha]N_\alpha] \subseteq N_{3\alpha}$$

by

$$(w_1 \wedge w_2)\Psi = [[w_1, w_2]v_1].$$

Let $\overline{\Phi}\colon W \wedge W \to N_{2\alpha}/M_{2\alpha}$ denote the composition of Φ with the canonical epimorphism and $\overline{\Psi}\colon W \wedge W \to N_{3\alpha}/M_{3\alpha}$ denote the composition of Ψ with the canonical epimorphism.

(4.6) Let $w \in W \wedge W$ and $e_1, e_2 \in V_1^\perp$. Then by the Jacobi identity

$$\left(w\Psi, \left[\left[e_1, d_{11}\right], e_2\right]\right) = \left(\left[w\Phi, v_1\right], \left[\left[e_1, d_{11}\right], e_2\right]\right) = A + B + D + E$$

where

$$A = \left(w\Phi, \left[v_1, \left[\left[e_1, d_{11}\right], e_2\right]\right]\right), \quad B = \left(\left[\left[w\Phi, \left[e_1, d_{11}\right]\right], e_2\right], v_1\right),$$
$$D = \left(\left[e_1, \left[d_{11}, \left[w\Phi, e_2\right]\right]\right], v_1\right), \quad E = \left(\left[\left[e_1, \left[w\Phi, e_2\right]\right], d_{11}\right], v_1\right).$$

Now

$$\left[v_1, \left[\left[e_1, d_{11}\right], e_2\right]\right] \in \left[L_\alpha\left[\left[M_\alpha^\perp, M_\alpha^\perp\right]M_\alpha^\perp\right]\right] \subseteq \left[M_\alpha^\perp, M_\alpha^\perp\right] \subseteq M_{2\alpha}^\perp$$

by Lemma 4.2. Thus $A \in (w\Phi, M_{2\alpha}^\perp)$.

Since $[w\Phi, [e_1, d_{11}]] \in \overline{H}$ we have

$$B \in \left(\left[\overline{H}, V_1^\perp\right], V_0\right) \subseteq (V_0^\perp, V_0) = (0) \quad \text{by (4.3.4)}.$$

Similarly $[d_{11}, [w\Phi, e_2]] \in \overline{H}$ so $D = 0$.

Finally, since (by (4.4.1))

$$\left[e_1, \left[w\Phi, e_2\right]\right] \in (e_1, \left[w\Phi, e_2\right])c_\alpha + \overline{H}_\alpha,$$

and since

$$\left(\left[\overline{H}_\alpha, d_{11}\right], v_1\right) \subseteq \left(\left[\overline{H}_\alpha, V_0^\perp\right], V_0\right) \subseteq (V_0^\perp, V_0) = (0)$$

(by (4.3.3)), while

$$(\left[c_\alpha, d_{11}\right], v_1) = -(d_{11}, \left[c_\alpha, v_1\right]) = -(v_1C, d_{11}) = -1,$$

we have $E = ([w\Phi, e_2], e_1)$.

Let $J = \{(s, j, r, i) \in \mathbf{Z}^4 | 1 < r \leqslant s < p - 1, 1 \leqslant i, j \leqslant n, \text{ and } i < j \text{ if } r = s\}$. Then $W \wedge W$ has basis $\{v_iC^r \wedge v_jC^s | (s, j, r, i) \in J\}$. Order J lexicographically. Then we have

(4.7) Lemma. *Let*

$$G = \left(\left[\left(v_{i'}C^{r'} \wedge v_{j'}C^{s'}\right)\Phi, d_{ir}\right], d_{j,s+1}\right)$$

where (s, j, r, i) *and* $(s', j', r', i') \in J$. *Then* $G = -1$ *if* $(s, j, r, i) = (s', j', r', i')$ *and* $G = 0$ *if* $(s, j, r, i) > (s', j', r', i')$.

Proof. By the Jacobi identity

$$G = \left(\left[\left[v_{i'}C^{r'}, v_{j'}C^{s'}\right], d_{ir}\right], d_{j,s+1}\right)$$
$$= \left(\left[\left[v_{i'}C^{r'}, d_{ir}\right], v_{j'}C^{s'}\right], d_{j,s+1}\right) + \left(\left[v_{i'}C^{r'}, \left[v_{j'}C^{s'}, d_{ir}\right]\right], d_{j,s+1}\right).$$

Assume $(s, j, r, i) \geqslant (s', j', r', i')$. Since (by (4.3.1)) $[\overline{H}_\alpha, v_{j'}C^{s'}] \subseteq V_{s'}$ and $[v_{i'}C^{r'}, \overline{H}_\alpha] \subseteq V_{r'} \subseteq V_{s'}$, while $d_{j,s+1} \in V_s^\perp \subseteq V_{s'}^\perp$, we see from (4.4.1) that

$$G = -\left(v_{i'}C^{r'}, d_{ir}\right)\left(v_{j'}C^{s'+1}, d_{j,s+1}\right) + \left(v_{j'}C^{s'}, d_{ir}\right)\left(v_{i'}C^{r'+1}, d_{j,s+1}\right).$$

The first summand is -1 if $(s, j, r, i) = (s', j', r', i')$ and is zero otherwise. If the second summand is nonzero then $s + 1 = r' + 1 \leqslant s' + 1 = r + 1$ so $r = s = r' = s'$. But then $j = i' < j' = i < j$, a contradiction. Thus the second summand is zero, proving the lemma.

(4.8) LEMMA. $\ker \overline{\Phi} \cap \ker \overline{\Psi} = (0)$.

PROOF. Suppose $w \in \ker \overline{\Phi} \cap \ker \overline{\Psi}$. Since $w\overline{\Psi} = 0$ we have $w\Psi \in M_{3\alpha}$, so by Lemma 4.2, if $e_1, e_2 \in V_1^{\perp}$, we have

$$\left(w\Psi, \left[\left[e_1, d_{11}\right], e_2\right]\right) \in (M_{3\alpha}, M_{3\alpha}^{\perp}) = (0).$$

Also, since $w\overline{\Phi} = 0$ we have $w\Phi \in M_{2\alpha}$. Thus in (4.6) $A \in (M_{2\alpha}, M_{2\alpha}^{\perp}) = (0)$. Thus

$$0 = \left(w\Psi, \left[\left[e_1, d_{11}\right], e_2\right]\right) = \left(\left[w\Phi, e_2\right], e_1\right).$$

If $w \neq 0$ then

$$w = u\left(v_i C^r \wedge v_j C^s\right) + \sum_{l \in J, l < (s,j,r,i)} w_l,$$

where $0 \neq u \in F$ and where w_l is a scalar multiple of the basis element corresponding to $l \in J$. But then Lemma 4.7 shows that $0 = [[w\Phi, d_{i,r}], d_{j,s+1}] = -u$. This contradiction shows $w = 0$.

(4.9) COROLLARY. $\dim N_{2\alpha}/M_{2\alpha} + \dim N_{3\alpha}/M_{3\alpha} > 2(\dim N_\alpha/M_\alpha)$.

PROOF. Since, by Lemma 4.8, $\overline{\Phi} \oplus \overline{\Psi}$ injects $W \wedge W$ into $N_{2\alpha}/M_{2\alpha} \oplus N_{3\alpha}/M_{3\alpha}$, it is sufficient to show that $\dim W \wedge W > 2(\dim N_\alpha/M_\alpha)$. Now we have $\dim N_\alpha/M_\alpha = np$ (by (4.3)) and $\dim W = n(p-3)$. Hence the corollary holds if $n(p-3)(n(p-3)-1)/2 > 2np$ or, equivalently, if $n(p-3)^2 > 5p - 3$. Since $p > 7$ this is valid for all $n \geqslant 1$.

(4.10) COROLLARY. $n_{2\alpha} + n_{3\alpha} > 2n_\alpha$.

PROOF. Apply Corollary 4.9 to each quotient in a composition series from K_α to L_α.

5. Dimension arguments. II.

(5.1) We continue to let F, L, and H be as in Theorem 2.1 and to assume $\overline{H} \neq T + I$. In addition, we assume that (3.3.2) holds. Our object is to show that $n_{\beta+\alpha} + n_{\beta-\alpha} > 2n_\beta$, contradicting (3.5.2).

(5.2) If $v \in L_\beta$ write $vC = [v, c_\beta]$. Let

$$L_\beta = W_t \supseteq W_{t-1} \supseteq \cdots \supseteq W_1 \supseteq W_0 = K_\beta,$$

where each W_i is a $\overline{H}$-submodule of L_β and each W_{i+1}/W_i is an irreducible $\overline{H}$-module. Let $W_{i,0} = \{w \in W_i | [b, w] \in W_{i-1}\}$ for $1 \leqslant i \leqslant t$ and $W_{i,j+1} = W_{i,j} + W_{i,j}C$ for $0 \leqslant j \leqslant p-2$. Then by Lemma 3.4 we have a chain of ad $\overline{H}_\beta$ invariant subspaces:

$$W_i = W_{i,p-1} \supseteq W_{i,p-2} \supseteq \cdots \supseteq W_{i,1} \supseteq W_{i,0} \supseteq W_{i-1}.$$

Furthermore, if $\{v_{i,j} + W_{i-1} | 1 \leq j \leq n_i\}$ is a basis for $W_{i,0}/W_{i-1}$, then

$$\{v_{i,j}C^k + W_{i-1} | 1 \leq j \leq n_i, 0 \leq k \leq p-1\}$$

is a basis for W_i/W_{i-1}, so that

$$\{v_{i,j}C^k + K_\beta | 1 \leq i \leq t, 1 \leq j \leq n_i, 0 \leq k \leq p-1\}$$

is a basis for L_β/K_β. Thus, if $n = \sum_{i=1}^{t} n_i$, we have $n_\beta = pn$. Finally, again by Lemma 3.4, we have

$$[W_{i,j}, \overline{H}_\beta] \subseteq W_{i,j} \quad \text{for all } i, j.$$

For $1 \leq i \leq t$, $1 \leq j \leq n_i$, $0 \leq k \leq p-1$ choose $d_{i,j,k} \in L_{-\beta}$ so that

$$(v_{r,s}C^q, d_{i,j,k}) = \delta_{ir}\delta_{js}\delta_{kq}.$$

(5.3) LEMMA. *There exist elements* $w_1, \ldots, w_p \in L_\alpha$ *and* $u_1, \ldots, u_p \in L_{-\alpha}$ *such that*

$$\beta([b[w_i, u_j]]) = \delta_{ij}$$

and, hence,

$$[w_i, u_j] \in \delta_{ij}c_\beta + \overline{H}_\beta.$$

PROOF. Let $Q_\alpha = \{x \in L_\alpha | \beta([b[x, L_{-\alpha}]]) = (0)\}$. Then Q_α is an $\overline{H}$-submodule of L_α. Applying Lemma 3.4 to an irreducible submodule of L_α/Q_α (which is nonzero since $(\beta, \alpha) \in S$) gives the result.

(5.4) Let J' denote $\{(r, i, j, k) | 1 \leq r \leq p, 1 \leq i \leq t, 1 \leq j \leq n_i, 0 \leq k \leq p-1\}$ and $J = \{(r, i, j, k) \in J' | k \neq p-1\}$.

LEMMA. *The* np^2 *by* np^2 *matrix*

$$\left(\left([[w_r, u_{r'}]v_{i,j}C^k], d_{i',j',k'}\right)\right)_{(r,i,j,k),(r',i',j',k') \in J'}$$

has rank $\geq n(p-1)p$.

PROOF. It is sufficient to show that the rows corresponding to $(r, i, j, k) \in J$ are linearly independent. Thus assume that for each $(r, i, j, k) \in J$ we have $a_{r,i,j,k} \in F$ such that

$$0 = \sum_J a_{r,i,j,k}\left([[w_r, u_{r'}]v_{i,j}C^k], d_{i',j',k'}\right)$$

for all $(r', i', j', k') \in J'$. We must show that all $a_{r,i,j,k} = 0$.

Assume that $1 \leq q \leq t$ and $0 \leq u \leq p-2$ and that $a_{r,i,j,k} = 0$ for all $(r, i, j, k) \in J$ with $i > q$ or $i = q$ and $k > u$. (This condition is vacuous if $q = t$, $u = p-2$.) We will show $a_{r,q,j,u} = 0$ for all r and j and, hence, by induction that $a_{r,i,j,k} = 0$ for all $(r, i, j, k) \in J$.

We have, for all r', $1 \leq r' \leq p$, and all j', $1 \leq j' \leq n_q$,

$$0 = \sum_J a_{r,i,j,k}\big([[w_r, u_{r'}]v_{i,j}C^k], d_{q,j',u+1}\big).$$

Since $a_{r,i,j,k} = 0$ if $i > q$ and

$$\big([[w_r, u_{r'}]v_{i,j}C^k], d_{q,j',u+1}\big) \in (W_i, d_{q,j',u+1}) = (0)$$

if $i < q$, we have

$$0 = \sum_{r,j,k} a_{r,q,j,k}\big([[w_r, u_{r'}]v_{q,j}C^k], d_{q,j',u+1}\big).$$

Since, by (5.2) and (5.3),

$$[[w_r, u_{r'}]v_{q,j}C^k] \in [\delta_{r,r'}c_\beta, v_{q,j}C^k] + [\overline{H}_\beta, W_{q,k}] \subseteq -\delta_{r,r'}v_{q,j}C^{k+1} + W_{q,k}$$

and since, by the induction assumption, $a_{r,q,j,k} = 0$ if $k > u$, we have

$$0 = -\sum_j a_{r',q,j,u}\big(v_{q,j}C^{u+1}, d_{q,j',u+1}\big) = -a_{r',q,j',u},$$

as required.

(5.5) Let $x_\alpha \in L_\alpha$, $x_\beta \in L_\beta$, $y_\alpha \in L_{-\alpha}$, and $y_\beta \in L_{-\beta}$. Then

$$\begin{aligned}
&([x_\alpha, x_\beta], [y_\alpha, y_\beta]) + ([x_\beta, y_\alpha], [x_\alpha, y_\beta]) \\
&\quad = ([[x_\alpha, x_\beta]y_\alpha], y_\beta) + (y_\alpha, [[x_\alpha, x_\beta]y_\beta]) \\
&\quad\quad + ([[x_\beta, y_\alpha]x_\alpha], y_\beta) + (x_\alpha, [[x_\beta, y_\alpha]y_\beta]) \\
&\quad = ([[x_\alpha, x_\beta]y_\alpha], y_\beta) + ([[x_\beta, y_\alpha]x_\alpha], y_\beta) \quad (\text{since } (\alpha, \alpha) \notin S) \\
&\quad = ([[x_\alpha, y_\alpha]x_\beta], y_\beta).
\end{aligned}$$

Now

$$n_{\beta+\alpha} = \dim(L_{\beta+\alpha}/K_{\beta+\alpha}) \geq \operatorname{rank}([x_\alpha, x_\beta], [y_\alpha, y_\beta])$$

where $x_\alpha, x_\beta, y_\alpha, y_\beta$ run over subsets of the appropriate L's. A similar remark holds for $n_{\beta-\alpha}$. Also, if A and B are matrices then

$$\operatorname{rank} A + \operatorname{rank} B \geq \operatorname{rank}(A + B).$$

Then setting $x_\alpha = w_r$, $y_\alpha = u_{r'}$, $x_\beta = v_{i,j}C^k$, and $y_\beta = d_{i',j',k'}$, where (r, i, j, k) and $(r', i', j', k') \in J'$, we see from Lemma 5.4 that $n_{\beta+\alpha} + n_{\beta-\alpha} \geq n(p - 1)p$. Since $p - 1 > 2$ we have $n_{\beta+\alpha} + n_{\beta-\alpha} > 2np = 2n_\beta$, as required.

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